資訊網路與多媒體研究所

資格考試科目：資訊理論與編碼技巧

99. 3. 12

Information Theory and Coding Techniques

1. Assume *f(x)* is a convex function and $\overline{X}$ is a set of random variables.
   Prove that $\sum_{i=1}^{k} p_i f(x_i) \geq f\left(\sum_{i=1}^{k} p_i x_i\right)$
   Where $x_i \in \overline{X}$ and p_i is the corresponding probability of x_i.

2. Prove the following Inequalities :
   (a) Let p(x), q(x), $x \in \overline{X}$, be two probability mass functions.
   Then $D(p//q) \geq 0$, with equality iff p(x)=q(x) for all x.(where iff means if and only if)
   (b) For any two random variables, X, Y,
   $I(X;Y) \geq 0$, with equality iff X and Y are independent.
   (c) $H(X) \leq \log|\overline{X}|$, where $|\overline{X}|$ denotes the number of elements in the range of $\overline{X}$, with equality iff X has a uniform distribution over $\overline{X}$.
   (d) $H(X|Y) \leq H(X) \leq H(X,Y)$
   (e) If $X \rightarrow Y \rightarrow Z$, then $I(X;Y) \geq I(X;Z)$

3. Prove the convexity and/or concavity of the following functions:
   (a) H(p) is a concave function of P.
   (b) D(p//q) is convex in the pair (p,q)
   (c) I(X;Y) is a concave function of p(x) for fixed p(y|x)
   (d) I(X;Y) is a convex function of p(y|x) for fixed p(x).

4. Source A ⟶ Encoder C ⟶

Source symbol a	Codeword b=C(a)
P(a), H(A)	P(b), H(B)
|A|:# of source symbols	|B|: # of codewords

   (a) Prove that we cannot build a lossless encoder with |B|<|A|.
   (b) Can we build an encoder with |B|>|A| and somehow obtain more information, i.e., H(B)>H(A), than was emitted by the source. If Yes? How? If No, Why?

5. Prove that
   (a) If $X_1, X_2, \ldots$ are i.i.d~p(x), then

   $$-\frac{1}{n}\log P(X_1, X_2, \ldots, X_n) \rightarrow H(X) \text{ in probability.}$$

   (b) For any non-negative random variable X and t>0,

   Show that $P_r\{X \geq t\} \leq \frac{EX}{t}$, where 'E' stands for the operator of "expectation".

(c) Let Y be a random variable with mean μ and variance σ^2.

By letting $X = (Y - \mu)^2$, show that for any $\epsilon > 0$,

$P_r\{|Y - \mu| > \epsilon\} \leq \frac{\sigma^2}{\epsilon^2}$.

6. (a) Please find the range of the expected code length under the source probability p(x) of the code assignment $l(x) = \left\lceil \log \frac{1}{q(x)} \right\rceil$, where $\lceil x \rceil$ stands for the largest integer smaller than or equal to X.

   (b) Let l(x) be the codeword lengths associated with the Shannon code, and l'(x) be the codeword lengths associated with any other uniquely decodable code. Then prove that

$$P_r(l(x) \geq l'(x) + c) \leq \frac{1}{2^{c-1}}$$

7. Consider the random variable

$$X = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ 0.49 & 0.26 & 0.12 & 0.04 & 0.04 & 0.03 & 0.02 \end{pmatrix}$$

   (a) Find a binary Huffman code for X
   (b) Find the expected code length for this encoding
   (c) Find a ternary Huffman code for X
   (d) Find the expected code length for the encoding (c).

8. Please draw the block diagrams for the following two coding standards:
   (a) JPEG Encoder and Decoder
   (b) MPEG-1 Encoder and Decoder
   (c) Explain briefly the function of each component in your diagrams of codecs.