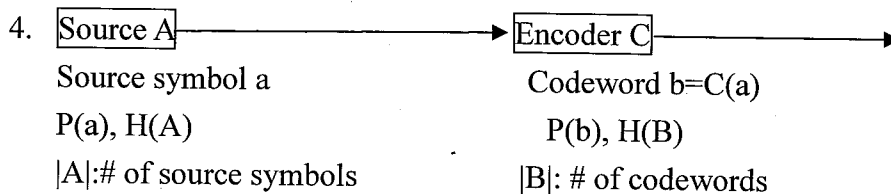


1. Assume $f(x)$ is a convex function and \bar{X} is a set of random variables.
 Prove that $\sum_{i=1}^k p_i f(x_i) \geq f(\sum_{i=1}^k p_i x_i)$
 Where $x_i \in \bar{X}$ and p_i is the corresponding probability of x_i .

2. Prove the following Inequalities :
 - (a) Let $p(x), q(x), x \in \bar{X}$, be two probability mass functions.
 Then $D(p//q) \geq 0$, with equality iff $p(x)=q(x)$ for all x .(where iff means if and only if)
 - (b) For any two random variables, X, Y ,
 $I(X;Y) \geq 0$, with equality iff X and Y are independent.
 - (c) $H(X) \leq \log|\bar{X}|$, where $|\bar{X}|$ denotes the number of elements in the range of \bar{X} ,
 with equality iff X has a uniform distribution over \bar{X} .
 - (d) $H(X|Y) \leq H(X) \leq H(X, Y)$
 - (e) If $X \rightarrow Y \rightarrow Z$, then $I(X; Y) \geq I(X; Z)$

3. Prove the convexity and/or concavity of the following functions:
 - (a) $H(p)$ is a concave function of P .
 - (b) $D(p//q)$ is convex in the pair (p,q)
 - (c) $I(X;Y)$ is a concave function of $p(x)$ for fixed $p(y|x)$
 - (d) $I(X;Y)$ is a convex function of $p(y|x)$ for fixed $p(x)$.



- (a) Prove that we cannot build a lossless encoder with $|B| < |A|$.
- (b) Can we build an encoder with $|B| > |A|$ and somehow obtain more information, i.e., $H(B) > H(A)$, than was emitted by the source. If Yes? How? If No, Why?

5. Prove that
 - (a) If X_1, X_2, \dots are i.i.d $\sim p(x)$, then

$$-\frac{1}{n} \log P(X_1, X_2, \dots, X_n) \rightarrow H(X)$$
 in probability.
 - (b) For any non-negative random variable X and $t > 0$,

Show that $P_r\{X \geq t\} \leq \frac{EX}{t}$, where 'E' stands for the operator of "expectation".

(c) Let Y be a random variable with mean μ and variance σ^2 .

By letting $X = (Y - \mu)^2$, show that for any $\epsilon > 0$,

$$P_r\{|Y - \mu| > \epsilon\} \leq \frac{\sigma^2}{\epsilon^2}.$$

6. (a) Please find the range of the expected code length under the source probability $p(x)$ of the code assignment $l(x) = \lceil \log_{q(x)} \frac{1}{p(x)} \rceil$, where $\lceil x \rceil$ stands for the largest integer smaller than or equal to X .

(b) Let $l(x)$ be the codeword lengths associated with the Shannon code, and $l'(x)$ be the codeword lengths associated with any other uniquely decodable code. Then prove that

$$P_r(l(x) \geq l'(x) + c) \leq \frac{1}{2^{c-1}}$$

7. Consider the random variable

$$X = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ 0.49 & 0.26 & 0.12 & 0.04 & 0.04 & 0.03 & 0.02 \end{pmatrix}$$

(a) Find a binary Huffman code for X

(b) Find the expected code length for this encoding

(c) Find a ternary Huffman code for X

(d) Find the expected code length for the encoding (c).

8. Please draw the block diagrams for the following two coding standards:

(a) JPEG Encoder and Decoder

(b) MPEG-1 Encoder and Decoder

(c) Explain briefly the function of each component in your diagrams of codecs.