Information Theory and Coding Techniques

Qualify Examination

1. Let X be a discrete random variable and g(x) is a function of x. please show that $H(g(x)) \leq H(x)$

2. A function $\rho(x,y)$ is a metric if for all x,y the following properties hold.
   (i) $\rho(x,y) \geq 0$, (ii) $\rho(x,y)=\rho(y,x)$, (iii) $\rho(x,y)=0$ iff x=y , and
   (iv) $\rho(x,y)+\rho(y,z)\geq \rho(x,z)$. Please show that
   (a) $\rho(X,Y)=H(X|Y)+H(Y|X)$ is a metric if X=Y (or there is a one-to-one function mapping from X to Y)
   (b) Verify that

$$\begin{aligned}\rho(X,Y)&=H(X)+H(Y)-2I(X;Y)\\&=H(X,Y)-I(X;Y)\\&=2H(X,Y)-H(X)-H(Y)\end{aligned}$$

3. Let p(x,y)be given by

X \ Y	0	1
0	$\frac{1}{3}$	$\frac{1}{3}$
1	0	$\frac{1}{3}$

Please find
   (a) $H(X), H(Y)$
   (b) $H(X|Y), H(Y|X)$
   (c) $H(X,Y)$
   (d) $I(X;Y)$

4. Let the random variable X have three possible outcomes $\{a, b, c\}$
   Consider two distributions on X:

symbo	p(x)	q(x)
a	$\frac{1}{2}$	$\frac{1}{3}$
b	$\frac{1}{4}$	$\frac{1}{3}$
c	$\frac{1}{4}$	$\frac{1}{3}$

Please find
   (a) $H(p), H(q)$
   (b) $D(p||q)$
   (c) $D(q||p)$

5. For the source $X=\begin{Bmatrix} x_1, & x_2, & x_3, & x_4, & x_5, & x_6 \\ 0.25, & 0.25, & 0.2, & 0.1, & 0.1, & 0.1 \end{Bmatrix}$

   Please construct an optimal Tenary Huffman code for X and find the corresponding average codeword length.

6. Which of the following codes are
   (a) Uniquely decodable?
   (b) Instantaneous decodable?

$C_1 = \{00, 01, 0\}$
$C_2 = \{00, 01, 100, 101, 11\}$
$C_3 = \{0, 10, 110, 1110, \ldots\}$
$C_4 = \{0, 00, 000, 0000\}$

7. Calculate the capacity of the following channels with probability transition matrices:
   (a) $X = Y = \{0, 1, 2\}$ $\quad P(y|x) = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$
   (b) $X = Y = \{0, 1, 2\}$ $\quad P(y|x) = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$
   (c) $X = Y = \{0, 1, 2, 3\}$ $\quad P(y|x) = \begin{bmatrix} P & 1-P & 0 & 0 \\ 1-P & P & 0 & 0 \\ 0 & 0 & q & 1-q \\ 0 & 0 & 1-q & q \end{bmatrix}$

8. Find an optimal set of binary codeword lengths $l_1, l_2, \ldots$ (minimizing $\sum P_i l_i$) for and instantaneous code for each of the following probability mass functions:
   (a) $P = \left(\frac{10}{41}, \frac{9}{41}, \frac{8}{41}, \frac{7}{41}, \frac{7}{41}\right)$
   (b) $P = \left(\frac{9}{10}, \left(\frac{9}{10}\right)\left(\frac{1}{10}\right), \left(\frac{9}{10}\right)\left(\frac{1}{10}\right)^2, \left(\frac{9}{10}\right)\left(\frac{1}{10}\right)^3, \ldots\right)$