

資訊網路與多媒體研究所

博士班資格考試題

科目：資訊理論與編碼技巧

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1. Prove that (H and D are the entropy and the relative entropy functions, and X, Y, Z are three different random variables, respectively.)

$$(1) H(X, Y | Z) = H(X | Z) + H(Y | X, Z).$$

$$(2) H(Y | X) \neq H(X | Y).$$

$$(3) H(X) - H(X | Y) = H(Y) - H(Y | X).$$

$$(4) D(p(x, y) \| q(x, y)) = D(p(x) \| q(x)) + D(p(y | x) \| q(y | x)).$$

2. Let a random variable $X = \begin{cases} 0, & p(0) = P \\ 1, & p(1) = 1 - P \end{cases}$,

where p is a probability function. Please prove the following:

$$(1) H(X) = 1 \text{ bits when } P = 1/2.$$

(2) $H(X)$ is a concave function of P .

(3) $H(X) = 0$ if $P = 0$ or 1 .

(4) The max $H(X)$ occurs when $P = 1/2$.

3. A function $\rho(x, y)$ is a metric if for all x, y ,

- $\rho(x, y) \geq 0$.
- $\rho(x, y) = \rho(y, x)$.
- $\rho(x, y) = 0$ if and only if $x = y$.
- $\rho(x, y) + \rho(y, z) \geq \rho(x, z)$.

(1) Show that $\rho(X, Y) = H(X | Y) + H(Y | X)$ satisfies the first, second, and fourth properties above. If we say that $X = Y$ if there is one-to-one function mapping X to Y , then the third property is also satisfied, and $\rho(X, Y)$ is a metric.

(2) Verify that $\rho(X, Y)$ can also be expressed as

$$\begin{aligned} \rho(X, Y) &= H(X) + H(Y) - 2I(X; Y) \\ &= H(X, Y) - I(X; Y) \\ &= 2H(X, Y) - H(X) - H(Y). \end{aligned}$$

4. Consider the following method for generating a code for a random variable X which takes on m values $\{1, 2, \dots, m\}$ with probabilities p_1, p_2, \dots, p_m . Assume that the probabilities are ordered so that $p_1 \geq p_2 \geq \dots \geq p_m$. Define

$$F_i = \sum_{k=1}^{i-1} p_k,$$

the sum of the probabilities of all symbols less than i . Then the codeword for i is

the number $F_i \in [0,1]$ rounded off to l_i bits, where $l_i = \left\lceil \log \frac{1}{p_i} \right\rceil$.

- (1) Show that the code constructed by this process is prefix-free and the average length satisfies

$$H(X) \leq L \leq H(X) + 1.$$

- (2) Construct the code for the probability distribution (0.5, 0.25, 0.125, 0.125).