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資訊網路與多媒體研究所

資格考試科目：資訊理論與編碼技巧

Information Theory and Coding Techniques

Qualify Examination

1. Let X be a discrete random variable and g(x) is a function of x. please show that $H(g(x)) \leq H(x)$

2. A function $\rho(x,y)$ is a metric if for all x,y the following properties hold.
   (i) $\rho(x,y) \geq 0$, (ii) $\rho(x,y)=\rho(y,x)$, (iii) $\rho(x,y)=0$ iff x=y , and
   (iv) $\rho(x,y)+\rho(y,z)\geq\rho(x,z)$. Please show that
   (a) $\rho(X,Y)=H(X|Y)+H(Y|X)$ is a metric if X=Y (or there is a one-to-one function mapping from X to Y)
   (b) Verify that

$$\begin{aligned}\rho(X,Y)&=H(X)+H(Y)-2I(X;Y)\\&=H(X,Y)-I(X;Y)\\&=2H(X,Y)-H(X)-H(Y)\end{aligned}$$

3. Let the random variable X have three possible outcomes $\{a, b, c\}$
   Consider two distributions on X:

symbo	p(x)	q(x)
a	$\frac{1}{2}$	$\frac{1}{3}$
b	$\frac{1}{4}$	$\frac{1}{3}$
c	$\frac{1}{4}$	$\frac{1}{3}$

4. For the source $X=\left\{\begin{matrix} x_1, & x_2, & x_3, & x_4, & x_5, & x_6 \\ 0.25, & 0.25, & 0.2, & 0.1, & 0.1, & 0.1 \end{matrix}\right\}$
   Please construct an optimal Tenary Huffman code for X and find the corresponding average codeword length.

5. Which of the following codes are
   (a) Uniquely decodable?
   (b) Instantaneous decodable?

$C_1 = \{00, 01, 0\}$
$C_2 = \{00, 01, 100, 101, 11\}$
$C_3 = \{0, 10, 110, 1110, \ldots\}$
$C_4 = \{0, 00, 000, 0000\}$

6. Calculate the capacity of the following channels with probability transition matrices:

(a) $X = Y = \{0, 1, 2\}$ $\quad P(y|x) = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$

(b) $X = Y = \{0, 1, 2\}$ $\quad P(y|x) = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$

(c) $X = Y = \{0, 1, 2, 3\}$ $\quad P(y|x) = \begin{bmatrix} P & 1-P & 0 & 0 \\ 1-P & P & 0 & 0 \\ 0 & 0 & q & 1-q \\ 0 & 0 & 1-q & q \end{bmatrix}$

7. *Cascade of binary symmetric channels.* Show that a cascade of n identical independent binary symmetric channels,

$$X_0 \rightarrow \boxed{\text{BSC}} \rightarrow X_1 \rightarrow \cdots \rightarrow X_{n-1} \rightarrow \boxed{\text{BSC}} \rightarrow X_n,$$

each with raw error probability p, is equivalent to a single BSC with error probability $\frac{1}{2}(1-(1-2p)^n)$ and hence that $\lim_{n\to\infty} I(X_0; X_n) = 0$ if $p \neq 0, 1$. No encoding or decoding takes place at the intermediate terminals $X_1, \ldots, X_{n-1}$. Thus, the capacity of the cascade tends to zero.

8. Let $l(x)$ be the codeword lengths associated with the Shannon code and let $l'(x)$ be the codeword lengths associated with any other codes.
Please show that

$$P_r(l(x) \geq l'(x) + c) \leq \frac{1}{2^{c-1}}$$