Information Theory and Coding Technique

1. Let X be a random variable taking on a finite number of values. What is the (general) inequality relationship of $H(X)$ and $H(Y)$ if
   (a) $Y = 2^X$?
   (b) $Y = \cos X$?
2. Let $X_1 \rightarrow X_2 \rightarrow \cdots \rightarrow X_n$ form a Markov chain in this order:
   That is, let $p(x_1, x_2, \ldots, x_n) = p(x_1)\, p(x_2|x_1) \ldots p(x_n|x_{n-1})$.
   Reduce $I(X_1; X_2, X_3, \ldots, X_n)$ to its simplest form.
3. The World Series is a seven-game series that terminates as soon as either team wins four games. Let X be the random variable that represents the outcome of a World Series between teams A and B; possible values of X are AAAA, BABABAB, and BBBAAAA. Let Y be the number of games played, which ranges from 4 to 7. Assuming that A and B are equally matched and that the games are independent, calculate $H(X)$, $H(Y)$, $H(Y|X)$, and $H(X|Y)$.
4. Let $X_1, X_2, \ldots, X_n$ be (possibly dependent) binary random variables. Suppose that are calculates the RUN LENGTHS $R = (R_1, R_2, \ldots)$ of this sequence (in order as they occur). For example, the sequence $X = 0001100100$ yields run lengths $R = (3,2,2,1,2)$.
   Compare $H(X_1, X_2, X_3, \ldots, X_n)$, $H(R)$, and $H(X_n, R)$.
   Show all equalities and inequalities, and bound all the differences.
5. Let X_i be i.i.d. ~ p(x), $x \in \{1, 2, \ldots, m\}$.
   Let $\mu = \mathrm{E}X$ and $H = -\sum p(x) \log p(x)$.

   Let $A^{\mathrm{n}} = \{x^n \in \aleph^n: \ | -\frac{1}{n} \log p(x^n) - H | \le \varepsilon \}$, and

   $B^{\mathrm{n}} = \{x^n \in \aleph^n: \ | \frac{1}{n} \sum_{i=1}^{n} X_i - \mu | \le \varepsilon \}$.
   (a) Does $\Pr\{x^n \in A^{\mathrm{n}}\} \rightarrow 1$?
   (b) Does $\Pr\{x^n \in A^{\mathrm{n}} \cap \boldsymbol{B}^{\mathrm{n}}\} \rightarrow 1$?
   (c) Show that $|A^{\mathrm{n}} \cap \boldsymbol{B}^{\mathrm{n}}| \leqq 2^{n(H+\varepsilon)}$ for all n.
   (d) Show that $|A^{\mathrm{n}} \cap \boldsymbol{B}^{\mathrm{n}}| \geqq (\frac{1}{2}) 2^{n(H-\varepsilon)}$ for n sufficiently large.