資訊網路與多媒體研究所

資格考試科目：資訊理論與編碼技巧

Information Theory and Coding Techniques

Qualify Examination 2012/9/28

1. Let X_1 and X_2 be identically distributed but not necessarily independent.

   Let $\varphi = 1 - \frac{H(X_2|X_1)}{H(X_1)}$,

   (a) Show that $\varphi = \frac{I(X_1;X_2)}{H(X_1)}$

   (b) Show that $0 \le \varphi \le 1$

   (c) When is $\varphi = 0$?

   (d) When is $\varphi = 1$?

2. Let $X_1 \to X_2 \to X_3 \to \cdots \to X_n$ form a Markov chain in this order; that is, let
   $p(x_1, x_2, \cdots, x_n) = p(x_1)p(x_2|x_1)\cdots p(x_n|x_{n-1})$
   Please reduce $I(X_1; X_2, \cdots X_n)$ to its simplest form.

3. Let X,Y,Z be three random variables with a joint probability mass function p(x,y,z).
   The relative entropy between the joint distribution and the product of marginal is

   $$D(p(x,y,z)\|p(x)p(y)p(z)) = E\left[\log \frac{p(x,y,z)}{p(x)p(y)p(z)}\right]$$

   (a) Expand this in terms of entropies.

   (b) When is this quantity zero?

4. Let $X = \begin{cases} 1, \text{with probability } \frac{1}{2} \\ 2, \text{with probability } \frac{1}{4} \\ 3, \text{with probability } \frac{1}{4} \end{cases}$

   Let X1,X2,… be drawn i.i.d. according to this distribution.

   Find the limiting behavior of the product

   $(X_1 \cdot X_2 \cdots X_n)^{\frac{1}{n}}$

5. Let A be a second-order Markov process with transition probabilities

   Pr(a=0|0,0)=0.2 Pr(a=1|0,0)=0.8

   Pr(a=0|0,1)=0.4 Pr(a=1|0,1)=0.6

   Pr(a=0|1,0)=0.0 Pr(a=1|1,0)=1.0

   Pr(a=0|1,1)=0.5 Pr(a=1|1,1)=0.5

and assume all states are equally probable at time t=0.

(a) What are the state probabilities at t=1?

(b) What is the steady-state probability distribution for this Markov source?

(c) Find the entropy rate for this Markov source.

6. Which of the following codeword lengths can be the word lengths of a 3-ary Huffman code, and which cannot? Why?

(a) (1,2,2,2,2)

(b) (2,2,2,2,2,2,2,2,3,3,3)

7. Suppose the codeword that we use to describe a random variable X~p(x) always starts with a symbol chosen from the set {A, B, C}, followed by binary digits {0,1}. Thus, we have a tenary code for the first symbol and binary thereafter. Give the optional uniquely decodable code (minimum expected number of symbols) for the probability distribution

$$P = \left(\frac{16}{69}.\frac{15}{69},\frac{12}{69},\frac{10}{69},\frac{8}{69},\frac{8}{69}\right)$$

8. For the source $X = \begin{Bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ 0.25 & 0.25 & 0.2 & 0.1 & 0.1 & 0.1 \end{Bmatrix}$

Please construct an optimal Tenary Huffman code for X and find the corresponding average codeword length.

9. *Cascade of binary symmetric channels.* Show that a cascade of *n* identical independent binary symmetric channels,

$$X_0 \rightarrow \boxed{\text{BSC}} \rightarrow X_1 \rightarrow \cdots \rightarrow X_{n-1} \rightarrow \boxed{\text{BSC}} \rightarrow X_n ,$$

each with raw error probability *p*, is equivalent to a single BSC with error probability $\frac{1}{2}(1-(1-2p)^n)$ and hence that $\lim_{n\to\infty} I(X_0;X_n) = 0$ if $p \neq 0, 1$. No encoding or decoding takes place at the intermediate terminals $X_1, \ldots, X_{n-1}$. Thus, the capacity of the cascade tends to zero.

10. Please find the differential entropy of a continuous Gaussian random variable X, where $X \sim \varphi(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/2\sigma^2}$