1. Assume $f(x)$ is a convex function and $\bar{X}$ is a set of random variables. 
Prove that $\sum_{i=1}^{k} p_i f(x_i) \geq f\left(\sum_{i=1}^{k} p_i x_i\right)$
Where $x_i \in \bar{X}$ and $p_i$ is the corresponding probability of $x_i$.

2. Prove the following Inequalities:
   (a) Let $p(x), q(x), x \in \bar{X}$, be two probability mass functions.
       Then $D(p//q) \geq 0$, with equality iff $p(x)=q(x)$ for all $x$ (where iff means if and
       only if)
   (b) For any two random variables, $X, Y$,
       $I(X;Y) \geq 0$, with equality iff $X$ and $Y$ are independent.
   (c) $H(X) = \log |\bar{X}|$, where $|\bar{X}|$ denotes the number of elements in the range of $\bar{X}$,
       with equality iff $X$ has a uniform distribution over $\bar{X}$.
   (d) $H(X|Y) \leq H(X) \leq H(X,Y)$
   (e) If $X \rightarrow Y \rightarrow Z$, then $I(X;Y) \geq I(X;Z)$

3. Prove the convexity and/or concavity of the following functions:
   (a) $H(p)$ is a concave function of $p$.
   (b) $D(p//q)$ is convex in the pair $(p,q)$
   (c) $I(X;Y)$ is a concave function of $p(x)$ for fixed $p(y|x)$
   (d) $I(X;Y)$ is a convex function of $p(y|x)$ for fixed $p(x)$.

4. **Source $A$** \[\rightarrow\] **Encoder $C$**

   Source symbol $a$ \[\rightarrow\] Codeword $b=C(a)$
   $P(a), H(A)$ \[\rightarrow\] $P(b), H(B)$
   $|A|$: # of source symbols \[\rightarrow\] $|B|$: # of codewords
   (a) Prove that we cannot build a lossless encoder with $|B|<|A|$.
   (b) Can we build an encoder with $|B|>|A|$ and somehow obtain more information,
       i.e., $H(B)>H(A)$, than was emitted by the source. If Yes? How? If No,
       Why?

5. Prove that
   (a) If $X_1, X_2, \ldots$ are i.i.d $\sim p(x)$, then
       
       $-\frac{1}{n} \log P(X_1, X_2, \ldots, X_n) \rightarrow H(X)$ in probability.

   (b) For any non-negative random variable $X$ and $t>0$,
       Show that $P_r\{X \geq t\} \leq \frac{EX}{t}$, where ‘E’ stands for the operator of “expectation”.


(c) Let $Y$ be a random variable with mean $\mu$ and variance $\sigma^2$.

By letting $X = (Y - \mu)^2$, show that for any $\epsilon > 0$,

$$P \{ |Y - \mu| > \epsilon \} \leq \frac{\sigma^2}{\epsilon^2}.$$

6. (a) Please find the range of the expected code length under the source probability $p(x)$ of the code assignment $l(x) = \left\lfloor \log \frac{1}{q(x)} \right\rfloor$, where $\lfloor x \rfloor$ stands for the largest integer smaller than or equal to $X$.

(b) Let $l(x)$ be the codeword lengths associated with the Shannon code, and $l'(x)$ be the codeword lengths associated with any other uniquely decodable code. Then prove that

$$P \{ l(x) \geq l'(x) + c \} \leq \frac{1}{2^{c-1}}.$$

7. Consider the random variable

$$X = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ 0.49 & 0.26 & 0.12 & 0.04 & 0.04 & 0.03 & 0.02 \end{pmatrix}$$

(a) Find a binary Huffman code for $X$

(b) Find the expected code length for this encoding

(c) Find a ternary Huffman code for $X$

(d) Find the expected code length for the encoding (c).

8. Please draw the block diagrams for the following two coding standards:

(a) JPEG Encoder and Decoder

(b) MPEG-1 Encoder and Decoder

(c) Explain briefly the function of each component in your diagrams of codecs.