Information Theory and Coding Techniques

Qualify Examination

1. Let X be a discrete random variable and g(x) is a function of x. Please show that \( H(g(x)) \leq H(x) \)

2. A function \( \rho(x,y) \) is a metric if for all \( x,y \) the following properties hold.
   (i) \( \rho(x, y) \geq 0 \), (ii) \( \rho(x, y) = \rho(y, x) \), (iii) \( \rho(x, y) = 0 \) iff \( x = y \), and
   (iv) \( \rho(x, y) + \rho(y, z) \geq \rho(x, z) \). Please show that
   (a) \( \rho(X, Y) = H(X|Y) + H(Y|X) \) is a metric if \( X=Y \) (or there is a one-to-one function mapping from X to Y)
   (b) Verify that
   \[
   \rho(X, Y) = H(X) + H(Y) - 2I(X;Y)
   = H(X, Y) - I(X;Y)
   = 2H(X, Y) - H(X) - H(Y)
   \]

3. Let \( p(x,y) \) be given by
   \[
   \begin{array}{c|cc}
   X & 0 & 1 \\
   \hline
   Y & 1 & 1 \\
   & 3 & 3 \\
   & 0 & 1 \\
   & 3 & 3 \\
   \end{array}
   \]
   Please find
   (a) \( H(X), H(Y) \)
   (b) \( H(X|Y), H(Y|X) \)
   (c) \( H(X, Y) \)
   (d) \( I(X;Y) \)

4. Let the random variable \( X \) have three possible outcomes \( \{a, b, c\} \). Consider two distributions on \( X \):

   - | symbo | p(x) | q(x) |
     - | a | \( \frac{1}{2} \) | \( \frac{1}{3} \)
     - | b | \( \frac{1}{2} \) | \( \frac{1}{3} \)
     - | c | \( \frac{1}{2} \) | \( \frac{1}{3} \)
   Please find
   (a) \( H(p), H(q) \)
   (b) \( D(p||q) \)
   (c) \( D(q||p) \)

5. For the source \( X = \{ x_1, x_2, x_3, x_4, x_5, x_6 \} \)
   \[
   \begin{pmatrix}
   0.25, & 0.25, & 0.2, & 0.1, & 0.1, & 0.1
   \end{pmatrix}
   \]
   Please construct an optimal Tenary Huffman code for \( X \) and find the corresponding average codeword length.
6. Which of the following codes are
   (a) Uniquely decodable?
   \[ C_1 = \{00, 01, 0\} \]
   (b) Instantaneous decodable?
   \[ C_2 = \{00, 01, 100, 101, 11\} \]
   \[ C_3 = \{0, 10, 110, 1110, \ldots\} \]
   \[ C_4 = \{0, 00, 000, 0000\} \]

7. Calculate the capacity of the following channels with probability transition matrices:
   (a) \( X = Y = \{0, 1, 2\} \quad P(y|x) = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \)
   (b) \( X = Y = \{0, 1, 2\} \quad P(y|x) = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \)
   (c) \( X = Y = \{0, 1, 2, 3\} \quad P(y|x) = \begin{bmatrix} P & 1-P & 0 & 0 \\ 1-P & P & 0 & 0 \\ 0 & 0 & q & 1-q \\ 0 & 0 & 1-q & q \end{bmatrix} \)

8. Find an optimal set of binary codeword lengths \( l_1, l_2, \ldots \) (minimizing \( \sum P l_i \)) for and instantaneous code for each of the following probability mass functions:
   (a) \( P = \left(\frac{10}{41}, \frac{9}{41}, \frac{8}{41}, \frac{7}{41}\right) \)
   (b) \( P = \left(\frac{9}{10}, \left(\frac{9}{10}\right)^2, \left(\frac{9}{10}\right)^3, \ldots\right) \)